

Recent progress on Monotone Operator Theory

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Abstract

In this paper, we survey recent progress on the theory of maximally monotone operators in general Banach space. We also extend various of the results and leave some open questions.

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1 Introduction

We assume throughout that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X , and that X and X^* are paired by $\langle \cdot, \cdot \rangle$. The *open unit ball* and *closed unit ball* in X is denoted respectively by $U_X := \{x \in X \mid \|x\| < 1\}$ and $B_X := \{x \in X \mid \|x\| \leq 1\}$, and $\mathbb{N} := \{1, 2, 3, \dots\}$.

We recall the following basic fact regarding the second dual ball:

Fact 1.1 (Goldstine) (See [45, Theorem 2.6.26] or [32, Theorem 3.27].) *The weak*-closure of B_X in X^{**} is $B_{X^{**}}$.*

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We say a net $(a_\alpha)_{\alpha \in \Gamma}$ in X is *eventually bounded* if there exist $\alpha_0 \in \Gamma$ and $M \geq 0$ such that

$$\|a_\alpha\| \leq M, \quad \forall \alpha \succeq_\Gamma \alpha_0.$$

We denote by \longrightarrow and \rightharpoonup_{w^*} the norm convergence and weak* convergence of nets, respectively.

1.1 Monotone operators

Let $A: X \rightrightarrows X^*$ be a *set-valued operator* (also known as a relation, point-to-set mapping or multifunction) from X to X^* , i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\text{gra } A := \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$ be the *graph* of A . The *domain* of A is $\text{dom } A := \{x \in X \mid Ax \neq \emptyset\}$ and $\text{ran } A := A(X)$ is the *range* of A .

Recall that A is *monotone* if

$$(1) \quad \langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*) \in \text{gra } A \ \forall (y, y^*) \in \text{gra } A,$$

and *maximally monotone* if A is monotone and A has no proper monotone extension (in the sense of graph inclusion). Let $A: X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say (x, x^*) is *monotonically related to* $\text{gra } A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \text{gra } A.$$

Monotone operators have frequently shown themselves to be a key class of objects in both modern Optimization and Analysis; see, e.g., [12, 13, 15], the books [4, 21, 28, 47, 59, 62, 54, 56, 79, 80, 81] and the references given therein.

We now introduce the four fundamental properties of maximally monotone operators that our paper focusses on.

Definition 1.2 *Let $A: X \rightrightarrows X^*$ be maximally monotone. Then four key properties of monotone operators are defined as follows.*

- (i) *A is of dense type or type (D) (1971, [36], [48] and [63, Theorem 9.5]) if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with*

$$\inf_{(a, a^*) \in \text{gra } A} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exist a bounded net $(a_\alpha, a_\alpha^)_{\alpha \in \Gamma}$ in $\text{gra } A$ such that $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ weak* \times strong converges to (x^{**}, x^*) .*

- (ii) *A is of type negative infimum (NI) (1996, [58]) if*

$$\inf_{(a, a^*) \in \text{gra } A} \langle a - x^{**}, a^* - x^* \rangle \leq 0, \quad \forall (x^{**}, x^*) \in X^{**} \times X^*.$$

- (iii) A is of type Fitzpatrick-Phelps (FP) (1992, [34]) if whenever U is an open convex subset of X^* such that $U \cap \text{ran } A \neq \emptyset$, $x^* \in U$, and $(x, x^*) \in X \times X^*$ is monotonically related to $\text{gra } A \cap (X \times U)$ it must follow that $(x, x^*) \in \text{gra } A$.
- (iv) A is of “Brønsted-Rockafellar” (BR) type (1999, [60]) if whenever $(x, x^*) \in X \times X^*$, $\alpha, \beta > 0$ and

$$\inf_{(a, a^*) \in \text{gra } A} \langle x - a, x^* - a^* \rangle > -\alpha\beta$$

then there exists $(b, b^*) \in \text{gra } A$ such that $\|x - b\| < \alpha$, $\|x^* - b^*\| < \beta$.

As is now known (see Corollary 3.12 and [61, 58, 43]), the first three properties coincide. This coincidence is central to many of our proofs. Fact 2.3 also shows us that every maximally monotone operator of type (D) is of type (BR). (The converse fails, see Example 2.12(xiii).) Moreover, in reflexive space every maximally monotone operator is of type (D), as is the subdifferential operator of every proper closed convex function on a Banach space.

While monotone operator theory is rather complete in reflexive space — and for type (D) operators in general space — the general situation is less clear [21, 15]. Hence our continuing interest in operators which are not of type (D). Not every maximally monotone operator is of type (BR) (see Example 2.12(v)).

We say a Banach space X is of *type (D)* [15] if every maximally monotone operator on X is of type (D). At present the only known type (D) spaces are the reflexive spaces; and our work here suggests that there are no non-reflexive type (D) spaces. In [21, Exercise 9.6.3, page 450] such spaces were called (NI) spaces and some potential non-reflexive examples were conjectured; all of which are ruled out by our more recent work. In [21, Theorem 9.7.9, page 458] a variety of the pleasant properties of type (D) spaces was listed. In Section 3.3 we briefly study a new dual class of (DV) spaces.

1.2 Convex analysis

As much as possible we adopt standard convex analysis notation. Given a subset C of X , $\text{int } C$ is the *interior* of C and \overline{C} is the *norm closure* of C . For the set $D \subseteq X^*$, \overline{D}^{w^*} is the weak* closure of D , and the norm \times weak* closure of $C \times D$ is $\overline{C \times D}^{\|\cdot\| \times w^*}$. The *indicator function* of C , written as ι_C , is defined at $x \in X$ by

$$(2) \quad \iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases}$$

The *support function* of C , written as σ_C , is defined by $\sigma_C(x^*) := \sup_{c \in C} \langle c, x^* \rangle$. For every $x \in X$, the *normal cone* operator of C at x is defined by $N_C(x) := \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) := \emptyset$, if $x \notin C$; the *tangent cone* operator of C at x is defined by $T_C(x) :=$

$\{x \in X \mid \sup_{x^* \in N_C(x)} \langle x, x^* \rangle \leq 0\}$, if $x \in C$; and $T_C(x) := \emptyset$, if $x \notin C$. The *hypertangent cone* of C at x , $H_C(x)$, coincides with the interior of $T_C(x)$ (see [20, 19]).

Let $f: X \rightarrow]-\infty, +\infty]$. Then $\text{dom } f := f^{-1}(\mathbb{R})$ is the *domain* of f , and $f^*: X^* \rightarrow [-\infty, +\infty]: x^* \mapsto \sup_{x \in X} (\langle x, x^* \rangle - f(x))$ is the *Fenchel conjugate* of f . The *epigraph* of f is $\text{epi } f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$. Let the net $(y_\alpha, y_\alpha^*)_{\alpha \in I}$ be in $X \times X^*$ and $(x^{**}, x^*) \in X^{**} \times X^*$. We write $(y_\alpha, y_\alpha^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)$ when (y_α, y_α^*) converges to (x^{**}, x^*) in the weak*-topology $\omega(X^{**}, X^*) \times \|\cdot\|$. We say f is *proper* if $\text{dom } f \neq \emptyset$. Let f be proper. The *subdifferential* of f is defined by

$$\partial f: X \rightrightarrows X^*: x \mapsto \{x^* \in X^* \mid (\forall y \in X) \langle y - x, x^* \rangle + f(x) \leq f(y)\}.$$

We denote by J the *duality map*, i.e., the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$ mapping X to X^* . Let $g: X \rightarrow]-\infty, +\infty]$. Then the *inf-convolution* $f \square g$ is the function defined on X by

$$f \square g: x \mapsto \inf_{y \in X} (f(y) + g(x - y)).$$

Let Y be another real Banach space and $F_1, F_2: X \times Y \rightarrow]-\infty, +\infty]$. Then the *partial inf-convolution* $F_1 \square_1 F_2$ is the function defined on $X \times Y$ by

$$F_1 \square_1 F_2: (x, y) \mapsto \inf_{u \in X} (F_1(u, y) + F_2(x - u, y)).$$

Then $F_1 \square_2 F_2$ is the function defined on $X \times Y$ by

$$F_1 \square_2 F_2: (x, y) \mapsto \inf_{v \in Y} (F_1(x, y - v) + F_2(x, v)).$$

1.3 Structure of the paper

The remainder of this paper is organized as follows. In Section 2, we construct maximally monotone operators that are not of Gossez's dense-type (D) in many nonreflexive spaces, and present many related examples such as operators not of type (BR).

In Section 3, we show that monotonicity of dense type (type (D)), negative infimum type and Fitzpatrick- Phelps type all coincide. We reprise two recent proofs—by Marques Alves/Svaiter and Simons—showing the important result that every maximally monotone operator of negative infimum type defined on a real Banach space is actually of dense type.

In Section 4, we consider the structure of maximally monotone operators in Banach space whose domains have nonempty interior and we present new and explicit structure formulas for such operators. In Section 5, we list some important recent characterizations of monotone linear relations, such as a complete generalization of the Brezis-Browder theorem in general Banach space. Finally, in Section 6, we mention some central open problems in Monotone Operator Theory.

2 Type (D) space

In this section, we construct maximally monotone operators that are not of Gossez's dense-type (D) in nearly all nonreflexive spaces. Many of these operators also fail to possess the Brønsted-Rockafellar (BR) property. Using these operators, we show that the partial inf-convolution of two BC-functions will not always be a BC-function. This provides a negative answer to a challenging question posed by Stephen Simons. Among other consequences, we deduce — in a uniform fashion — that every Banach space which contains an isomorphic copy of the James space \mathbf{J} or its dual \mathbf{J}^* , or c_0 or its dual ℓ^1 , admits a non type (D) operator. The existence of non type (D) operators in spaces containing ℓ^1 or c_0 has been proved recently by Bueno and Svaiter.

This section is based on the work in [6] by Bauschke, Borwein, Wang and Yao.

Let $A : X \rightrightarrows X^*$ be linear relation. We say that A is *skew* if $\text{gra } A \subseteq \text{gra } (-A^*)$; equivalently, if $\langle x, x^* \rangle = 0$, $\forall (x, x^*) \in \text{gra } A$. Furthermore, A is *symmetric* if $\text{gra } A \subseteq \text{gra } A^*$; equivalently, if $\langle x, y^* \rangle = \langle y, x^* \rangle$, $\forall (x, x^*), (y, y^*) \in \text{gra } A$. We define the *symmetric part* and the *skew part* of A via

$$(3) \quad P := \frac{1}{2}A + \frac{1}{2}A^* \quad \text{and} \quad S := \frac{1}{2}A - \frac{1}{2}A^*,$$

respectively. It is easy to check that P is symmetric and that S is skew. Let S be a subspace of X . We say A is *S-saturated* [62] if

$$Ax + S^\perp = Ax, \quad \forall x \in \text{dom } A.$$

We say a maximally monotone operator $A : X \rightrightarrows X^*$ is *unique* if all maximally monotone extensions of A (in the sense of graph inclusion) in $X^{**} \times X^*$ coincide. Let Y be another real Banach space. We set $P_X : X \times Y \rightarrow X : (x, y) \mapsto x$, and $P_Y : X \times Y \rightarrow Y : (x, y) \mapsto y$. Let $L : X \rightarrow Y$ be linear. We say L is a (linear) *isomorphism* into Y if L is one to one, continuous and L^{-1} is continuous on $\text{ran } L$. We say L is an *isometry* if $\|Lx\| = \|x\|$, $\forall x \in X$. The spaces X, Y are then *isometric* (*isomorphic*) if there exists an isometry (*isomorphism*) from X onto Y .

Now let $F : X \times X^* \rightarrow]-\infty, +\infty]$. We say F is a *BC-function* (BC stands for “Bigger conjugate”) [62] if F is proper and convex with

$$(4) \quad F^*(x^*, x) \geq F(x, x^*) \geq \langle x, x^* \rangle \quad \forall (x, x^*) \in X \times X^*.$$

2.1 Operators of type (BR)

We first describe some properties of type (BR) operators. Let $A : X \rightrightarrows X^*$ be a maximally monotone operator. We say A is *isomorphically of type (BR)*, or *(BRI)* if, A is of type (BR) in every equivalent norm on X . Let us emphasize that we do not know if there exists a maximally monotone operator of type (BR) that is not isomorphically of type (BR). Note that all the other properties studied in this paper are preserved by Banach space isomorphism.

To produce operators not of type (D) but that are of type (BR) we exploit:

Lemma 2.1 (See [6, Lemma 3.2].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone and linear skew operator. Assume that $\text{gra}(-A^*) \cap X \times X^* \subseteq \text{gra } A$. Then A is isomorphically of type (BR).*

Lemma 2.1 shows that every continuous monotone linear and skew operator is of type (BR).

Corollary 2.2 (See [6, Corollary 3.3].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone and linear skew operator that is not of type (D). Assume that A is unique. Then $\text{gra } A = \text{gra}(-A^*) \cap X \times X^*$ and so A is isomorphically of type (BR).*

Fact 2.3 (Marques Alves and Svaiter) (See [42, Theorem 1.4(4)] or [41].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator that is of type (NI) (or equivalently, by Theorem 3.9, of type (D)). Then A is isomorphically of type (BR).*

Remark 2.4 Since (NI) is an isomorphic notion, by Fact 2.3, every operator of type (NI) is isomorphically of type (BR).

The next result will allow us to show that not every continuous monotone linear operator is of type (BR) (see Remark 2.13 below).

Proposition 2.5 *Let $A : X \rightrightarrows X^*$ be maximally monotone. Assume that there exists $e \in X^*$ such that $e \notin \overline{\text{ran } A}$ and that*

$$\langle a^*, a \rangle \geq \langle e, a \rangle^2, \quad \forall (a, a^*) \in \text{gra } A.$$

Then A is not of type (BR), and $P_{X^}[\text{dom } F_A] \not\subseteq \overline{\text{ran } A}$.*

Proof. Let $(x_0, x_0^*) := (0, e)$. Then we have

$$\begin{aligned} & \inf_{(a, a^*) \in \text{gra } A} (\langle a - x_0, a^* - x_0^* \rangle) = \inf_{(a, a^*) \in \text{gra } A} (\langle a, a^* - e \rangle) = \inf_{(a, a^*) \in \text{gra } A} (\langle a, a^* \rangle - \langle a, e \rangle) \\ (5) \quad & \geq \inf_{(a, a^*) \in \text{gra } A} (\langle a, e \rangle^2 - \langle a, e \rangle) \geq \inf_{t \in \mathbb{R}} (t^2 - t) = -\frac{1}{4}. \end{aligned}$$

Suppose to the contrary that A is of type (BR). Then Fact 4.21 implies that $e \in \overline{\text{ran } A}$, which contradicts the assumption that $e \notin \overline{\text{ran } A}$. Hence A is not of type (BR). By (5), $(0, e) \in \text{dom } F_A$ and $e \notin \overline{\text{ran } A}$. Hence $P_{X^*}[\text{dom } F_A] \not\subseteq \overline{\text{ran } A}$. ■

2.2 Operators of type (D)

We now turn to type (D) operators.

Fact 2.6 (Simons) (See [62, Theorem 28.9].) *Let Y be a Banach space, and $L : Y \rightarrow X$ be continuous and linear with $\text{ran } L$ closed and $\text{ran } L^* = Y^*$. Let $A : X \rightrightarrows X^*$ be monotone with $\text{dom } A \subseteq \text{ran } L$ such that $\text{gra } A \neq \emptyset$. Then A is maximally monotone if, and only if A is $\text{ran } L$ -saturated and L^*AL is maximally monotone.*

Fact 2.6 leads us to the following result.

Theorem 2.7 (See [6, Theorem 2.17].) *Let Y be a Banach space, and $L : Y \rightarrow X$ be an isomorphism into X . Let $T : Y \rightrightarrows Y^*$ be monotone. Then T is maximally monotone if, and only if $(L^*)^{-1}TL^{-1}$, mapping X into X^* , is maximally monotone.*

The following consequence will allow us to construct maximally monotone operators that are not of type (D) in a very wide variety of non-reflexive Banach spaces.

Corollary 2.8 (Subspaces) (See [6, Corollary 2.18].) *Let Y be a Banach space, and $L : Y \rightarrow X$ be an isomorphism into X . Let $T : Y \rightrightarrows Y^*$ be monotone. The following hold.*

- (i) *Assume that $(L^*)^{-1}TL^{-1}$ is maximally monotone of type (D). Then T is maximally monotone of type (D). In particular, every Banach subspace of a type (D) space is of type (D).*
- (ii) *If T is maximally monotone and not of type (D), then $(L^*)^{-1}TL^{-1}$ is a maximally monotone operator mapping X into X^* that is not of type (D).*

Remark 2.9 Note that it follows that X is of type (D) whenever X^{**} is. The necessary part of Theorem 2.7 was proved by Bueno and Svaiter in [27, Lemma 3.1]. A similar result to Corollary 2.8(i) was also obtained by Bueno and Svaiter in [27, Lemma 3.1] with the additional assumption that T be maximally monotone.

Theorem 2.10 below allows us to construct various maximally monotone operators — both linear and nonlinear — that are not of type (D). The idea of constructing the operators in the following fashion is based upon [2, Theorem 5.1] and was stimulated by [27].

Theorem 2.10 (Predual constructions) (See [6, Theorem 3.7].) *Let $A : X^* \rightarrow X^{**}$ be linear and continuous. Assume that $\text{ran } A \subseteq X$ and that there exists $e \in X^{**} \setminus X$ such that*

$$\langle Ax^*, x^* \rangle = \langle e, x^* \rangle^2, \quad \forall x^* \in X^*.$$

Let P and S respectively be the symmetric part and antisymmetric part of A . Let $T : X \rightrightarrows X^$ be defined by*

$$(6) \quad \text{gra } T := \{(-Sx^*, x^*) \mid x^* \in X^*, \langle e, x^* \rangle = 0\} = \{(-Ax^*, x^*) \mid x^* \in X^*, \langle e, x^* \rangle = 0\}.$$

Let $f : X \rightarrow]-\infty, +\infty]$ be a proper lower semicontinuous and convex function. Set $F := f \oplus f^$ on $X \times X^*$. Then the following hold.*

- (i) *A is a maximally monotone operator on X^* that is neither of type (D) nor unique.*
- (ii) *$Px^* = \langle x^*, e \rangle e$, $\forall x^* \in X^*$.*
- (iii) *T is maximally monotone and skew on X .*

- (iv) $\text{gra } T^* = \{(Sx^* + re, x^*) \mid x^* \in X^*, r \in \mathbb{R}\}.$
- (v) $-T$ is not maximally monotone.
- (vi) T is not of type (D).
- (vii) $F_T = \iota_C$, where $C := \{(-Ax^*, x^*) \mid x^* \in X^*\}.$
- (viii) T is not unique.
- (ix) T is not of type (BR).
- (x) If $\text{dom } T \cap \text{int dom } \partial f \neq \emptyset$, then $T + \partial f$ is maximally monotone.
- (xi) F and F_T are BC-functions on $X \times X^*.$
- (xii) Moreover,

$$\bigcup_{\lambda > 0} \lambda(P_{X^*}(\text{dom } F_T) - P_{X^*}(\text{dom } F)) = X^*,$$

while, assuming that there exists $(v_0, v_0^*) \in X \times X^*$ such that

$$(7) \quad f^*(v_0^*) + f^{**}(v_0 - A^*v_0^*) < \langle v_0, v_0^* \rangle,$$

then $F_T \square_1 F$ is not a BC-function.

- (xiii) Assume that $[\text{ran } A - \bigcup_{\lambda > 0} \lambda \text{dom } f]$ is a closed subspace of X and that

$$\emptyset \neq \text{dom } f^{**} \circ A^*|_{X^*} \not\subseteq \{e\}_\perp.$$

Then $T + \partial f$ is not of type (D).

- (xiv) Assume that $\text{dom } f^{**} = X^{**}.$ Then $T + \partial f$ is a maximally monotone operator that is not of type (D).

Remark 2.11 Let A be defined as in Theorem 2.10 By Proposition 2.5, A is not of type (BR) and then Fact 2.3 implies that A is not of type (D). Moreover, $P_{X^*}[\text{dom } F_A] \not\subseteq \overline{\text{ran } A}.$

The first application of this result is to $c_0.$

Example 2.12 (c_0) (See [6, Example 4.1].) Let $X := c_0$, with norm $\|\cdot\|_\infty$ so that $X^* = \ell^1$ with norm $\|\cdot\|_1$, and $X^{**} = \ell^\infty$ with its second dual norm $\|\cdot\|_*.$ Fix $\alpha := (\alpha_n)_{n \in \mathbb{N}} \in \ell^\infty$ with $\limsup \alpha_n \neq 0$, and let $A_\alpha : \ell^1 \rightarrow \ell^\infty$ be defined by

$$(8) \quad (A_\alpha x^*)_n := \alpha_n^2 x_n^* + 2 \sum_{i > n} \alpha_n \alpha_i x_i^*, \quad \forall x^* = (x_n^*)_{n \in \mathbb{N}} \in \ell^1.$$

Now let P_α and S_α respectively be the symmetric part and antisymmetric part of A_α . Let $T_\alpha : c_0 \rightrightarrows X^*$ be defined by

$$(9) \quad \begin{aligned} \text{gra } T_\alpha &:= \{(-S_\alpha x^*, x^*) \mid x^* \in X^*, \langle \alpha, x^* \rangle = 0\} = \{(-A_\alpha x^*, x^*) \mid x^* \in X^*, \langle \alpha, x^* \rangle = 0\} \\ &= \left\{ \left(-\sum_{i>n} \alpha_n \alpha_i x_i^* + \sum_{i<n} \alpha_n \alpha_i x_i^* \right)_{n \in \mathbb{N}}, x^* \mid x^* \in X^*, \langle \alpha, x^* \rangle = 0 \right\}. \end{aligned}$$

Then

- (i) $\langle A_\alpha x^*, x^* \rangle = \langle \alpha, x^* \rangle^2$, $\forall x^* = (x_n^*)_{n \in \mathbb{N}} \in \ell^1$ and (9) is well defined.
- (ii) A_α is a maximally monotone operator on ℓ^1 that is neither of type (D) nor unique.
- (iii) T_α is a maximally monotone operator on c_0 that is not of type (D). Hence c_0 is not of type (D).
- (iv) $-T_\alpha$ is not maximally monotone.
- (v) T_α is neither unique nor of type (BR).
- (vi) $F_{T_\alpha} \square_1(\|\cdot\| \oplus \iota_{B_{X^*}})$ is not a BC-function.
- (vii) $T_\alpha + \partial \|\cdot\|$ is a maximally monotone operator on $c_0(\mathbb{N})$ that is not of type (D).
- (viii) If $\frac{1}{\sqrt{2}} < \|\alpha\|_* \leq 1$, then $F_{T_\alpha} \square_1(\frac{1}{2}\|\cdot\|^2 \oplus \frac{1}{2}\|\cdot\|_1^2)$ is not a BC-function.
- (ix) For $\lambda > 0$, $T_\alpha + \lambda J$ is a maximally monotone operator on c_0 that is not of type (D).
- (x) Let $\lambda > 0$ and a linear isometry L mapping c_0 to a subspace of $C[0, 1]$ be given. Then both $(L^*)^{-1}(T_\alpha + \partial \|\cdot\|)L^{-1}$ and $(L^*)^{-1}(T_\alpha + \lambda J)L^{-1}$ are maximally monotone operators that are not of type (D). Hence $C[0, 1]$ is not of type (D).
- (xi) Every Banach space that contains an isomorphic copy of c_0 is not of type (D).
- (xii) Let $G : \ell^1 \rightarrow \ell^\infty$ be *Gossez's operator* [37] defined by

$$(G(x^*))_n := \sum_{i>n} x_i^* - \sum_{i<n} x_i^*, \quad \forall (x_n^*)_{n \in \mathbb{N}} \in \ell^1.$$

Then $T_e : c_0 \rightrightarrows \ell^1$ as defined by

$$\text{gra } T_e := \{(-G(x^*), x^*) \mid x^* \in \ell^1, \langle x^*, e \rangle = 0\}$$

is a maximally monotone operator that is not of type (D), where $e := (1, 1, \dots, 1, \dots)$.

- (xiii) Moreover, G is a unique maximally monotone operator that is not of type (D), but G is isomorphically of type (BR). \diamond

Remark 2.13 Let A_α be defined as in Example 2.12. By Remark 2.11, A_α is not of type (BR) and $P_{X^*}[\text{dom } F_{A_\alpha}] \not\subseteq \overline{\text{ran } A}$.

Remark 2.14 The maximal monotonicity of the operator T_e in Example 2.12(xii) was also verified by Voisei and Zălinescu in [74, Example 19] and later a direct proof was given by Bueno and Svaiter in [27, Lemma 2.1]. Herein we have given a new proof of the above results.

Bueno and Svaiter had already showed that T_e is not of type (D) in [27]. They also showed that each Banach space that contains an isometric (isomorphic) copy of c_0 is not of type (D) in [27]. Example 2.12(xi) recaptures their result, while Example 2.12(vi)&(viii) provide a negative answer to Simons' [62, Problem 22.12]. \diamond

In our earlier work we were able to especially exploit some properties of the *quasi-reflexive* James space (\mathbf{J} is of codimension one in \mathbf{J}^{**}):

Definition 2.15 *The James space, \mathbf{J} , consists of all the sequences $x = (x_n)_{n \in \mathbb{N}}$ in c_0 with the finite norm*

$$\|x\| := \sup_{n_1 < \dots < n_k} \left((x_{n_1} - x_{n_2})^2 + (x_{n_2} - x_{n_3})^2 + \dots + (x_{n_{k-1}} - x_{n_k})^2 \right)^{\frac{1}{2}}.$$

Corollary 2.16 (Higher duals) (See [6, Corollary 4.14].) *Suppose that both X and X^* admit maximally monotone operators not of type (D) then so does every higher dual space X^n . In particular, this applies to both $X = c_0$ and $X = \mathbf{J}$.*

3 Equivalence of three types of monotone operators

We now show that the three monotonicities notions, of dense type (type (D)), negative infimum type and Fitzpatrick-Phelps type all coincide.

3.1 Type (NI) implies type (D)

We reprise two recent proofs—by Marques Alves–Svaiter and by Simons—showing that every maximally monotone operator of negative infimum type defined on a real Banach space is actually of dense type. We do this since the result is now central to current research and deserves being made as accessible as possible.

The key to establishing (NI) implies (D) is in connecting the second conjugate of a convex function to the original function. The next Fact is well known in various forms, but since we work with many dual spaces, care has to be taken in writing down the details.

Fact 3.1 (Marques Alves and Svaiter) (See [43, Lemma 4.1].) *Let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous convex. Then*

$$(10) \quad F^{**}(x^{**}, x^*) = \liminf_{(y, y^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)} F(y, y^*), \quad \forall (x^{**}, x^*) \in X^{**} \times X^*.$$

Proof. Suppose that (10) fails. Then there exists $(x^{**}, x^*) \in X^{**} \times X^*$ such that

$$(11) \quad F^{**}(x^{**}, x^*) \neq \liminf_{(y, y^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)} F(y, y^*).$$

Since $F^{**} = F$ on $X \times X^*$, then

$$F^{**}(x^{**}, x^*) \leq \liminf_{(y, y^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)} F^{**}(y, y^*) = \liminf_{(y, y^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)} F(y, y^*),$$

and by (11),

$$(12) \quad F^{**}(x^{**}, x^*) < \liminf_{(y, y^*) \xrightarrow{w^{**} \times \|\cdot\|} (x^{**}, x^*)} F(y, y^*).$$

Hence, $((x^{**}, x^*), F^{**}(x^{**}, x^*)) \notin \overline{\text{epi } F}^{w^{**} \times \|\cdot\| \times \|\cdot\|}$, where $\overline{\text{epi } F}^{w^{**} \times \|\cdot\| \times \|\cdot\|}$ is the weak*-topology $\omega(X^{**}, X^*) \times \|\cdot\| \times \|\cdot\|$ in $X^{**} \times X^* \times \mathbb{R}$. By the Hahn-Banach Separation theorem, there exist $((z^*, z^{**}), \lambda) \in X^* \times X^{**} \times \mathbb{R}$ and $r \in \mathbb{R}$ such that

$$(13) \quad \langle y, z^* \rangle + \langle y^*, z^{**} \rangle + \langle F(y, y^*) + t, -\lambda \rangle < r < \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle + \langle F^{**}(x^{**}, x^*), -\lambda \rangle,$$

for every $(y, y^*) \in \text{dom } F$ and $t \geq 0$. Hence $\lambda \geq 0$. Next we show $\lambda \neq 0$. Suppose to the contrary that $\lambda = 0$. Then (13) implies that

$$(14) \quad \sup_{(y, y^*) \in \text{dom } F} \langle y, z^* \rangle + \langle y^*, z^{**} \rangle \leq r < \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle.$$

As in [46, Section 6], $\text{dom } F^{**}$ is a subset of the closure of $\text{dom } F$ in the topology $\omega(X^{**}, X^*) \times \omega(X^{***}, X^{**})$, and so

$$\sup_{(y, y^*) \in \text{dom } F} \langle y, z^* \rangle + \langle y^*, z^{**} \rangle \geq \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle$$

since $(x^{**}, x^*) \in \text{dom } F^{**}$ by (12). This contradicts (14). Hence $\lambda \neq 0$ and so $\lambda > 0$.

Taking $t = 0$ and the supremum on (y, y^*) in (13), we deduce

$$\lambda F^*\left(\frac{z^*}{\lambda}, \frac{z^{**}}{\lambda}\right) = (\lambda F)^*(z^*, z^{**}) < \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle - \lambda F^{**}(x^{**}, x^*),$$

and so we have

$$(15) \quad \lambda F^*\left(\frac{z^*}{\lambda}, \frac{z^{**}}{\lambda}\right) + \lambda F^{**}(x^{**}, x^*) < \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle.$$

But by Fenchel-Young inequality [21],

$$\lambda F^*(\frac{z^*}{\lambda}, \frac{z^{**}}{\lambda}) + \lambda F^{**}(x^{**}, x^*) \geq \lambda(\langle \frac{z^*}{\lambda}, x^{**} \rangle + \langle \frac{z^{**}}{\lambda}, x^* \rangle) = \langle x^{**}, z^* \rangle + \langle x^*, z^{**} \rangle,$$

which contradicts (15). Hence (10) holds. ■

Let $CLB(X)$ denote the set of all convex functions from X to \mathbb{R} that are Lipschitz on the bounded subsets of X , and $T_{CLB}(X^{**})$ on X^{**} be the topology on X^{**} such that h^{**} is continuous everywhere on X^{**} for every $h \in CLB(X)$. (For more information about $CLB(X)$ and $T_{CLB}(X^{**})$ see [62].)

This prepares us for the main tool in the proof of Theorem 3.2. We describe two proofs, the first is due to Simons, and the second is due to Marques Alves and Svaiter. The goal is to restrict ourselves to bounded nets.

Theorem 3.2 (Marques Alves and Svaiter) (See [43, Theorem 4.2].) *Let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be proper, (norm) lower semicontinuous and convex. Consider $(x^{**}, x^*) \in X^{**} \times X^*$. Then there exists a bounded net $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ in $X \times X^*$ that converges to (x^{**}, x^*) in the weak*-topology $\omega(X^{**}, X^*) \times \|\cdot\|$ such that*

$$F^{**}(x^{**}, x^*) = \lim F(x_\alpha, x_\alpha^*).$$

Proof. The first proof, due to Simons, is very concise but quite abstract.

Proof one: By [62, Lemma 45.9(a)], there exists a net $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ in $X \times X^*$ that converges to (x^{**}, x^*) in the $T_{CLB}(X^{**} \times X^{**})$ such that

$$F^{**}(x^{**}, x^*) = \lim F(x_\alpha, x_\alpha^*).$$

By [62, Lemma 49.1(b)] (apply X^* to H) (or [63, Lemma 7.3(b)]), $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ converges to (x^{**}, x^*) in the $T_{CLB}(X^{**}) \times \|\cdot\|$. Then by [63, Lemma 7.1(a)], $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ is eventually bounded and converges to x^{**} in the weak*-topology $\omega(X^{**}, X^*) \times \|\cdot\|$.

The second proof, due to Marques Alvez and Svaiter, is equally concise but more direct.

Proof two: We consider two cases.

Case 1: $(x^{**}, x^*) \in \text{dom } F^{**}$. Fix $M > 0$ such that $\|(x^{**}, x^*)\| \leq M$ and $\text{dom } F \cap \text{int } MB_{X \times X^*} \neq \emptyset$, where $B_{X \times X^*}$ is a closed unit ball of $X \times X^*$. Then as in the proof of [53, Eqn. (2.8) of Prop. 1], [21, Theorem 4.4.18] or [43, Lemma 2.3],

$$\begin{aligned} (F + \iota_{MB_{X \times X^*}})^{**}(x^{**}, x^*) &= \left[F^* \square \iota_{MB_{X \times X^*}}^* \right]^*(x^{**}, x^*) = (F^{**} + \iota_{MB_{X \times X^*}}^{**})(x^{**}, x^*) \\ &= F^{**}(x^{**}, x^*). \end{aligned}$$

Finally, we may directly apply Fact 3.1 with $F + \iota_{MB_{X \times X^*}}$ replacing F .

Case 2: $(x^{**}, x^*) \notin \text{dom } F^{**}$. Then $F^{**}(x^{**}, x^*) = +\infty$. By Goldstine's theorem, there exists a bounded net $(x_\alpha)_{\alpha \in I}$ in X such that $(x_\alpha)_{\alpha \in I}$ converges to x^{**} in the weak*-topology $\omega(X^{**}, X^*)$.

Take $x_\alpha^* = x^*, \forall \alpha \in I$. By lower semicontinuity of F^{**} and since $F^{**} = F$ on $X \times X^*$, we have $\lim F(x_\alpha, x_\alpha^*) = +\infty$, and hence we have $F^{**}(x^{**}, x^*) = \lim F(x_\alpha, x_\alpha^*)$ as asserted. \blacksquare

Similar arguments to those of the first proof of Theorem 3.2 lead to:

Proposition 3.3 *Let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be proper, lower semicontinuous and convex. Let $(x, x^{***}) \in X \times X^{***}$. Then there exists a bounded net $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ in $X \times X^*$ that converges to (x, x^{***}) in the norm \times weak*-topology $\omega(X^{***}, X^{**})$ such that*

$$F^{**}(x, x^{***}) = \lim F(x_\alpha, x_\alpha^*).$$

We now apply Theorem 3.2 to *representative* functions attached to A ; more precisely, to the *Fitzpatrick function*:

$$(16) \quad F_A(x, x^*) := \sup_{a^* \in Aa} \left(\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle \right).$$

Let Y be another real Banach space and $F : X \times Y \rightarrow]-\infty, +\infty]$. We define $F^\Gamma : Y \times X \rightarrow]-\infty, +\infty]$ by

$$F^\Gamma(y, x) := F(x, y), \quad \forall (x, y) \in X \times Y.$$

Three more building blocks follow:

Fact 3.4 (Fitzpatrick) (See ([33, Propositions 3.2&4.1, Theorem 3.4].) *Let Y be a real Banach space and $A : Y \rightrightarrows Y^*$ be monotone with $\text{dom } A \neq \emptyset$. Then F_A is proper lower semicontinuous, convex, and $(F_A)^{\Gamma} = F_A = \langle \cdot, \cdot \rangle$ on $\text{gra } A$.*

Fact 3.5 (Fitzpatrick) (See [33, Theorem 3.10].) *Let $A : X \rightrightarrows X^*$ be maximally monotone, and let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be convex. Assume that $F \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$ and $F = \langle \cdot, \cdot \rangle$ on $\text{gra } A$. Then $F_A \leq F$.*

Fact 3.6 (Simons) (See [58, Theorem 19(c)].) *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (NI). Then the operator $B : X^* \rightrightarrows X^{**}$ defined by*

$$(17) \quad \text{gra } B := \{ (x^*, x^{**}) \in X^* \times X^{**} \mid \langle x^* - a^*, x^{**} - a \rangle \geq 0, \forall (a, a^*) \in \text{gra } A \}$$

is maximally monotone.

We may now establish an important link between A and B :

Proposition 3.7 *Let $A : X \rightrightarrows X^*$ be maximally monotone of type (NI), and let the operator $B : X^* \rightrightarrows X^{**}$ be defined as in (17). Then $(F_B)^\Gamma \leq (F_A)^{**} \leq (F_B)^*$ on $X^{**} \times X^*$. In consequence, $(F_A)^{**} = \langle \cdot, \cdot \rangle$ on $\text{gra } B$.*

Proof. Define $F : X \times X^* \rightarrow]-\infty, +\infty]$ by

$$(y, y^*) \mapsto \langle y, y^* \rangle + \iota_{\text{gra } A}(y, y^*).$$

We have

$$\begin{aligned} F^*(x^*, x^{**}) &= \sup_{(a, a^*) \in \text{gra } A} \{ \langle x^*, a \rangle + \langle x^{**}, a^* \rangle - \langle a, a^* \rangle \} \\ (18) \quad &= \langle x^*, x^{**} \rangle - \inf_{(a, a^*) \in \text{gra } A} \langle x^{**} - a, x^* - a^* \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}. \end{aligned}$$

Since A is of type (NI), $\inf_{(a, a^*) \in \text{gra } A} \langle x^{**} - a, x^* - a^* \rangle \leq 0$ and so (18) shows

$$(19) \quad F^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**}.$$

Now when $(x^*, x^{**}) \in \text{gra } B$, then $\inf_{(a, a^*) \in \text{gra } A} \langle x^{**} - a, x^* - a^* \rangle \geq 0$. Hence, $F^*(x^*, x^{**}) \leq \langle x^*, x^{**} \rangle$ by (18) again. Combining this with (19), leads to $F^*(x^*, x^{**}) = \langle x^*, x^{**} \rangle$. Thus, $F^* \geq \langle \cdot, \cdot \rangle$ on $X^* \times X^{**}$ and $F^* = \langle \cdot, \cdot \rangle$ on $\text{gra } B$. Thence, Fact 3.5 and Fact 3.6 imply that

$$(20) \quad F^* \geq F_B.$$

As $F_A = F$ on $\text{dom } F = \text{gra } A$, by Fact 3.5 we have $F_A \leq F$ and so $(F_A)^* \geq F^*$. Hence, by (20),

$$(21) \quad (F_A)^* \geq F_B.$$

By Fact 3.4, $(F_A)^* = \langle \cdot, \cdot \rangle$ on $\text{gra } A^{-1}$ and so we have $H := \langle \cdot, \cdot \rangle + \iota_{\text{gra } A^{-1}} \geq (F_A)^*$ on $X^* \times X^{**}$. Thus by (21),

$$(22) \quad H \geq (F_A)^* \geq F_B \quad \text{on } X^* \times X^{**}.$$

Fix $(x^*, x^{**}) \in X^* \times X^{**}$. On conjugating we obtain $F^*(x^*, x^{**}) = H^*(x^{**}, x^*) \leq (F_A)^{**}(x^{**}, x^*) \leq (F_B)^*(x^{**}, x^*)$ and so by (20),

$$(23) \quad F_B(x^*, x^{**}) \leq (F_A)^{**}(x^{**}, x^*) \leq (F_B)^*(x^{**}, x^*)$$

To conclude the proof, if $(x^*, x^{**}) \in \text{gra } B$, then by Fact 3.4 (applied in $Y = X^*$ on appealing to Fact 3.6) $F_B(x^*, x^{**}) = \langle x^*, x^{**} \rangle = (F_B)^*(x^{**}, x^*)$. Then $(F_A)^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$ and as asserted $(F_A)^{**} = \langle \cdot, \cdot \rangle$ on $\text{gra } B$. \blacksquare

Fact 3.8 (Voisei and Zălinescu) (See [73, Theorem 2.6] and [43, Theorem 4.4], [63, Theorem 6.12(a) and Remark 5.13] or [21, Theorem 9.7.2].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator that is of type (NI). Then*

$$F_A(x, x^*) - \langle x, x^* \rangle \geq \frac{1}{2} \inf_{(a, a^*) \in \text{gra } A} \{ \|x - a\|^2 + \|x^* - a^*\|^2 \} \geq \frac{1}{4} \inf_{(a, a^*) \in \text{gra } A} \{ \|x - a\|^2 + \|x^* - a^*\|^2 \}.$$

The proof of Theorem 3.9 now follows the lines of that of [63, Theorem 6.15 (c) \Rightarrow (a)].

Theorem 3.9 (Marques Alves and Svaiter) (See [43, Theorem 4.4] or [63, Theorem 9.5].) *Let $A : X \rightrightarrows X^*$ be maximally monotone. Assume that A is of type (NI). Then A is of type (D).*

Proof. Let $(x^{**}, x^*) \in X^{**} \times X^*$ be monotonically related to $\text{gra } A$, and let B be defined by (17) of Fact 3.6. Then $(x^*, x^{**}) \in \text{gra } B$. Thus by Proposition 3.7, $(F_A)^{**}(x^{**}, x^*) = \langle x^{**}, x^* \rangle$. By Theorem 3.2, there exists a bounded net $(x_\alpha, x_\alpha^*)_{\alpha \in I}$ in $X \times X^*$ that converges to (x^{**}, x^*) in the weak*-topology $\omega(X^{**}, X^*) \times \|\cdot\|$ such that

$$\langle x^{**}, x^* \rangle = (F_A)^{**}(x^{**}, x^*) = \lim F_A(x_\alpha, x_\alpha^*).$$

By Fact 3.4,

$$\langle x^{**}, x^* \rangle = \lim F_A(x_\alpha, x_\alpha^*) \geq \lim \langle x_\alpha, x_\alpha^* \rangle = \langle x^{**}, x^* \rangle.$$

Then we have $0 \leq \varepsilon_\alpha := F_A(x_\alpha, x_\alpha^*) - \langle x_\alpha, x_\alpha^* \rangle \rightarrow 0$. By Fact 3.8, there exists a net $(a_\alpha, a_\alpha^*)_{\alpha \in I}$ in $\text{gra } A$ with

$$\|a_\alpha - x_\alpha\| \leq 2\sqrt{\varepsilon_\alpha} \quad \text{and} \quad \|a_\alpha^* - x_\alpha^*\| \leq 2\sqrt{\varepsilon_\alpha}.$$

So, $(a_\alpha, a_\alpha^*)_{\alpha \in I}$ is eventually bounded and converges to (x^{**}, x^*) in the topology $\omega(X^{**}, X^*) \times \|\cdot\|$ as required. ■

3.2 Type (FP) implies type (NI)

This section relies on work in [7] by Bauschke, Borwein, Wang and Yao.

Fact 3.10 (Simons) (See [61, Theorem 17] or [62, Theorem 37.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone and of type (D). Then A is of type (FP).*

Theorem 3.11 (See [7, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone such that A is of type (FP). Then A is of type (NI).*

We now obtain the promised corollary:

Corollary 3.12 (See [7, Corollary 3.2].) *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then the following are equivalent.*

- (i) A is of type (D).
- (ii) A is of type (NI).
- (iii) A is of type (FP).

Proof. “(i) \Rightarrow (iii)”: Fact 3.10. “(iii) \Rightarrow (ii)”: Theorem 3.11 . “(ii) \Rightarrow (i)”: Theorem 3.9. ■

We turn to the construction of linear and nonlinear maximally monotone operators that are of type (BR) but not of type (D).

Proposition 3.13 *Let $A : X \rightrightarrows X^*$ be maximally monotone, and let Y be another real Banach space and $B : Y \rightrightarrows Y^*$ be maximally monotone of type (BR). Assume that A is of type (BR) but not of type (D). Define the norm on $X \times Y$ by $\|(x, y)\| := \|x\| + \|y\|$. Let $T : X \times Y \rightrightarrows X^* \times Y^*$ be defined by $T(x, y) := (Ax, By)$. Then T is a maximally monotone operator that is of type (BR) but not of type (D). In consequence, if A and B are actually isomorphically (BR) then so is T .*

Proof. We first show that T is maximally monotone. Clearly, T is monotone. Let $((x, y), (x^*, y^*)) \in (X \times Y) \times (X^* \times Y^*)$ be monotonically related to $\text{gra } T$. Then we have

$$\langle x - a, x^* - a^* \rangle + \langle y - b, y^* - b^* \rangle \geq 0, \quad \forall (a, a^*) \in \text{gra } A, (b, b^*) \in \text{gra } B.$$

Thus

$$(24) \quad \inf_{(a, a^*) \in \text{gra } A} \langle x - a, x^* - a^* \rangle + \inf_{(b, b^*) \in \text{gra } B} \langle y - b, y^* - b^* \rangle \geq 0.$$

Let $r := \inf_{(a, a^*) \in \text{gra } A} \langle x - a, x^* - a^* \rangle$. We consider two cases.

Case 1: $r \geq 0$. By maximal monotonicity of A , we have $(x, x^*) \in \text{gra } A$. Then $r = 0$. Thus by (24) and maximal monotonicity of B , $(y, y^*) \in \text{gra } B$ and hence $((x, y), (x^*, y^*)) \in \text{gra } T$.

Case 2: $r < 0$. Thus by (24)

$$(25) \quad \inf_{(b, b^*) \in \text{gra } B} \langle y - b, y^* - b^* \rangle \geq -r > 0.$$

Since B is maximally monotone, $(y, y^*) \in \text{gra } B$ and hence

$$\inf_{(b, b^*) \in \text{gra } B} \langle y - b, y^* - b^* \rangle = 0,$$

which contradicts (25).

Combining cases, we see that T is maximally monotone.

Next we show T is of type (BR) Let $((u, v), (u^*, v^*)) \in (X \times Y) \times (X^* \times Y^*)$, $\alpha, \beta > 0$ such that

$$(26) \quad \inf_{(a, a^*) \in \text{gra } A} \langle u - a, u^* - a^* \rangle + \inf_{(b, b^*) \in \text{gra } B} \langle v - b, v^* - b^* \rangle = \inf_{(z, z^*) \in \text{gra } T} \langle (u, v) - z, (u^*, v^*) - z^* \rangle > -\alpha\beta.$$

Let $-\rho = \inf_{(b, b^*) \in \text{gra } B} \langle v - b, v^* - b^* \rangle$. Then $\rho \geq 0$. We consider two cases.

Case 1: $\rho = 0$. Then $(v, v^*) \in \text{gra } B$. By (26) and since A is of type (BR), there exists $(a_1, a_1^*) \in \text{gra } A$ such that

$$\|u - a_1\| < \alpha, \quad \|u^* - a_1^*\| < \beta.$$

Thus

$$\|(u, v) - (a_1, v)\| = \|u - a_1\| < \alpha, \quad \|(u^*, v^*) - (a_1^*, v^*)\| = \|u^* - a_1^*\| < \beta.$$

Case 2: $\rho > 0$. By (26),

$$(27) \quad \inf_{(a, a^*) \in \text{gra } A} \langle u - a, u^* - a^* \rangle > -\alpha\beta + \rho = -(\alpha - \frac{\rho}{\beta})\beta.$$

Then by the maximal monotonicity of A ,

$$\inf_{(a, a^*) \in \text{gra } A} \langle u - a, u^* - a^* \rangle \leq 0.$$

Then by (27),

$$(28) \quad \alpha - \frac{\rho}{\beta} > 0.$$

Then by (27) and by that A is of type (BR), there exists $(a_2, a_2^*) \in \text{gra } A$ such that

$$(29) \quad \|u - a_2\| < \alpha - \frac{\rho}{\beta}, \quad \|u^* - a_2^*\| < \beta.$$

Since B is of type (BR), there exists $(b_2, b_2^*) \in \text{gra } B$ such that

$$(30) \quad \|v - b_2\| < \frac{\rho}{\beta}, \quad \|v^* - b_2^*\| < \beta.$$

Taking $(z_1, z_1^*) = ((a_2, b_2), (a_2^*, b_2^*))$ and combining (29) and (30), we have $(z_1, z_1^*) \in \text{gra } T$

$$\|(u, v) - z_1\| < \alpha, \quad \|(u^*, v^*) - z_1^*\| < \beta.$$

Hence T is of type (BR).

Lastly, we show T is not of type (D).

Since A is not of type (D), by Corollary 3.12, A is not of type (NI) and then there exists $(x_0^{**}, x_0^*) \in X^{**} \times X^*$ such that

$$(31) \quad \inf_{(a, a^*) \in \text{gra } A} \langle x_0^{**} - a, x_0^* - a^* \rangle > 0.$$

Take $(y_0, y_0^*) \in \text{gra } B$ and $(z_0^{**}, z_0^*) = (x_0^{**} + y_0, x_0^* + y_0^*)$. Then by (31),

$$\inf_{(z, z^*) \in \text{gra } T} \langle z_0^{**} - z, z_0^* - z^* \rangle = \inf_{(a, a^*) \in \text{gra } A} \langle x_0^{**} - a, x_0^* - a^* \rangle > 0.$$

Hence T is not of type (NI) and hence T is not of type (D) by Corollary 3.12.

When A and B are actually isomorphically (BR), following the above proof, we see that T is isomorphically (BR). ■

Corollary 3.14 *Let $B : \ell^2 \rightrightarrows \ell^2$ be an arbitrary maximally monotone operator and define $T : \ell^1 \times \ell^2 \rightrightarrows \ell^1 \times \ell^2$ by $T(x, y) := (G(x), B(x))$, where G is the Gossez operator. Then T is a maximally monotone operator that is of type (BR) isomorphically but not of type (D).*

Proof. Since ℓ^2 is reflexive, B is of type (BR). Then apply Example 2.12(xiii) and Proposition 3.13 directly. \blacksquare

Remark 3.15 In the case that A in Proposition 3.13 is nonaffine we obtain nonaffine operators of type (BR) which do not have unique extensions to the bidual, since, unless the operator is affine, uniqueness implies type (D) [42].

In the next section, we will explore properties of type (DV) operators as defined below. This a useful dual notion to type (D).

3.3 Properties of type (DV) operators

Let $A : X \rightrightarrows X^*$ be (maximally) monotone. We say A is of type Fitzpatrick-Phelps-Veronas (FPV) if for every open convex set $U \subseteq X$ such that $U \cap \text{dom } A \neq \emptyset$, the implication

$$x \in U \text{ and } (x, x^*) \text{ is monotonically related to } \text{gra } A \cap (U \times X^*) \Rightarrow (x, x^*) \in \text{gra } A$$

holds.

We also introduce a dual definition of type (DV), corresponding to the definition of type (D).

Definition 3.16 *Let $A : X \rightrightarrows X^*$ be maximally monotone. We say A is of type (DV) if for every $(x, x^{***}) \in X \times X^{***}$ with*

$$\inf_{(a, a^*) \in \text{gra } A} \langle a - x, a^* - x^{***} \rangle \geq 0,$$

there exists a bounded net $(a_\alpha, a_\alpha^)_{\alpha \in \Gamma}$ in $\text{gra } A$ such that $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ converges to (x, x^{***}) in the norm \times the weak*-topology $\omega(X^{***}, X^{**})$.*

Proposition 3.17 *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator. Let $B : X \rightrightarrows X^{***}$ be defined by $\text{gra } B := \{(a, a^{***}) \mid (a, a^{***}|_X) \in \text{gra } A\}$. Then B is a unique maximally monotone extension of $\text{gra } A$ in $X \times X^{***}$.*

Proof. Clearly, B is monotone with respect to $X \times X^{***}$. Now we show that B is maximally monotone with respect to $X \times X^{***}$.

Let $(x, x^{***}) \in X \times X^{***}$ be monotonically related to $\text{gra } A$. Let $x^* := X^{***}|_X$. Then $x^* \in X^*$ such that

$$\langle a - x, a^* - x^* \rangle = \langle a - x, a^* - x^{***} \rangle \geq 0, \quad \forall (a, a^*) \in \text{gra } A.$$

Since A is maximally monotone, $(x, x^*) \in \text{gra } A$ and then $(x, x^{***}|_X) \in \text{gra } A$. Hence $(x, x^{***}) \in \text{gra } B$. Thus, $\text{gra } B$ contains all the elements in $X \times X^{***}$ that are monotonically related to $\text{gra } A$. Since $\text{gra } B$ is monotone, B is a unique maximally monotone extension of A in $X \times X^{***}$. ■

The next result will confirm that type (DV) and type (D) are distinct. Neither every subdifferential operator nor every linear continuous monotone operator is of type (DV). In consequence, type (D) operators are not always type (DV).

Proposition 3.18 *Let $A : X \rightarrow X^*$ be a continuous maximally monotone operator. Then A is of type (DV) if and only if X is reflexive.*

Proof. “ \Leftarrow ”: Clear.

“ \Rightarrow ”: Suppose to the contrary that X is not reflexive. Since $X \subsetneq X^{**}$ and X is a closed subspace of X^{**} , by the Hahn-Banach theorem, there exists $x_0^{***} \in X^{***} \setminus \{0\}$ such that $\langle x_0^{***}, X \rangle = \{0\}$. Then we have

$$\langle a - 0, Aa - (A0 + x_0^{***}) \rangle = \langle a - 0, Aa - A0 \rangle - \langle a, x_0^{***} \rangle = \langle a - 0, Aa - A0 \rangle \geq 0, \quad \forall a \in X.$$

Then $(0, A0 + x_0^{***})$ is monotonically related to $\text{gra } A$. Since A is of type (DV), there exists a bounded net $(a_\alpha, Aa_\alpha)_{\alpha \in \Gamma}$ in $\text{gra } A$ such that $(a_\alpha, Aa_\alpha)_{\alpha \in \Gamma}$ converges to $(0, A0 + x_0^{***})$ in the norm \times the weak*-topology $\omega(X^{***}, X^{**})$. Since A is continuous and $a_\alpha \rightarrow 0$, we have $Aa_\alpha \rightarrow A0$ in X^* and hence $Aa_\alpha \rightarrow A0$ in X^{***} . Since Aa_α converges $A0 + x_0^{***}$ in the weak*-topology $\omega(X^{***}, X^{**})$, we have $A0 = A0 + x_0^{***}$ and hence $x_0^{***} = 0$, which contradicts that $x_0^{***} \in X^{***} \setminus \{0\}$. Hence X is reflexive. ■

Remark 3.19 Let P_α be defined in Example 2.12, then P_α is a subdifferential operator defined on ℓ^1 and also a bounded continuous linear operator. Then it is of type (D) but it is not of type (DV) by Proposition 3.18. Hence type (D) cannot imply type (DV).

Remark 3.20 It is unknown whether every maximal monotone operator is of type (FPV). Perhaps property (DV) may help shed light on the matter. We have also been unable to determine if (DV) implies (FPV).

We might say a Banach space X is of type (DV) if every maximally monotone operator on X is necessarily of type (DV).

Theorem 3.21 *The Banach space X is of type (DV) if and only if it is reflexive.*

Proof. “ \Leftarrow ”: Clear.

“ \Rightarrow ”: Let $A : X \rightarrow X$ defined by $\text{gra } A := X \times \{0\}$. Then A is maximally monotone continuous linear operator. Since A is of type (DV), Proposition 3.18 implies that X is reflexive. ■

Finally we give an example of a type (DV) operator in an arbitrary Banach space.

Example 3.22 Let X be a Banach space and let $A : X \rightrightarrows X^*$ be defined by $\text{gra } A := \{0\} \times X^*$. Let B be defined by $\text{gra } B := \{0\} \times X^{***}$. Then B is a unique maximally monotone extension of A in $X \times X^{***}$, and A is of type (DV). \diamond

Proof. By Proposition 3.17, B is a unique maximally monotone extension of A in $X \times X^{***}$. Then applying Goldstine's theorem (see Fact 1.1), A is of type (DV). \blacksquare

4 Structure of maximally monotone operators

We turn to the structure of maximally monotone operators in Banach space whose domains have nonempty interior and we present new and explicit structure formulas for such operators. Along the way, we provide new proofs of norm-to-weak* closedness and of property (Q) for these operators (as recently proven by Voisei). Various applications and limiting examples are given.

This section is mainly based on the work in [22, 23].

4.1 Local boundedness properties

The next two important results now have many proofs (see also [21, Ch. 8]).

Fact 4.1 (Rockafellar) (See [53, Theorem A], [79, Theorem 3.2.8], [62, Theorem 18.7] or [21, Theorem 9.2.1].) *Let $f : X \rightarrow]-\infty, +\infty]$ be a proper lower semicontinuous convex function. Then ∂f is maximally monotone.*

The prior result can fail in both incomplete normed spaces and in complete metrizable locally convex spaces [21].

Fact 4.2 (Rockafellar) (See [52, Theorem 1] or [47, Theorem 2.28].) *Let $A : X \rightrightarrows X^*$ be monotone with $\text{int dom } A \neq \emptyset$. Then A is locally bounded at $x \in \text{int dom } A$, i.e., there exist $\delta > 0$ and $K > 0$ such that*

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \text{dom } A.$$

Based on Fact 4.2, we can develop a stronger result.

Lemma 4.3 (Strong directional boundedness) (See [22, Lemma 4.1].) *Let $A : X \rightrightarrows X^*$ be monotone and $x \in \text{int dom } A$. Then there exist $\delta > 0$ and $M > 0$ such that $x + 2\delta B_X \subseteq \text{dom } A$ and $\sup_{a \in x + 2\delta B_X} \|Aa\| \leq M$. Assume also that (x_0, x_0^*) is monotonically related to $\text{gra } A$. Then*

$$\sup_{a \in [x + \delta B_X, x_0[, a^* \in Aa} \|a^*\| \leq \frac{1}{\delta} (\|x_0 - x\| + 1) (\|x_0^*\| + 2M),$$

where $[x + \delta B_X, x_0[:= \{(1 - t)y + tx_0 \mid 0 \leq t < 1, y \in x + \delta B_X\}$.

The following result — originally conjectured by the first author in [14] — was established by Voisei in [72, Theorem 37] as part of a more complex set of results.

Theorem 4.4 (Eventual boundedness) (See [22, Theorem 4.1].) *Let $A : X \rightrightarrows X^*$ be monotone such that $\text{int dom } A \neq \emptyset$. Then every $\text{norm} \times \text{weak}^*$ convergent net in $\text{gra } A$ is eventually bounded.*

Corollary 4.5 (Norm-weak* closed graph) (See [22, Corollary 4.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone such that $\text{int dom } A \neq \emptyset$. Then $\text{gra } A$ is $\text{norm} \times \text{weak}^*$ closed.*

Example 4.6 (Failure of graph to be norm-weak* closed) In [16], the authors showed that the statement of Corollary 4.5 cannot hold without the assumption of the nonempty interior domain even for the subdifferential operators — actually it fails in the bw^* topology. More precisely (see [16] or [4, Example 21.5]): Let $f : \ell^2(\mathbb{N}) \rightarrow]-\infty, +\infty]$ be defined by

$$(32) \quad x \mapsto \max \{1 + \langle x, e_1 \rangle, \sup_{2 \leq n \in \mathbb{N}} \langle x, \sqrt{n} e_n \rangle\},$$

where $e_n := (0, \dots, 0, 1, 0, \dots, 0)$: the n th entry is 1 and the others are 0. Then f is proper lower semicontinuous and convex, but ∂f is not $\text{norm} \times \text{weak}^*$ closed. A more general construction in an infinite-dimensional Banach space E is also given in [16, Section 3]. It is as follows:

Let Y be an infinite dimensional separable subspace of E , and $(v_n)_{n \in \mathbb{N}}$ be a *normalized Markushevich basis* of Y with the dual coefficients $(v_n^*)_{n \in \mathbb{N}}$. We defined $v_{p,m}$ and $v_{p,m}^*$ by

$$v_{p,m} := \frac{1}{p}(v_p + v_{p^m}) \quad \text{and} \quad v_{p,m}^* := v_p^* + (p - 1)v_{p^m}^*, \quad m \in \mathbb{N}, p \text{ is prime.}$$

Let $f : E \rightarrow]-\infty, +\infty]$ be defined by

$$(33) \quad x \mapsto \iota_Y(x) + \max \{1 + \langle x, v_1^* \rangle, \sup_{2 \leq m \in \mathbb{N}, p \text{ is prime}} \langle x, v_{p,m}^* \rangle\}.$$

Then f is proper lower semicontinuous and convex. We have that ∂f is not $\text{norm} \times \text{bw}^*$ closed and hence ∂f is not $\text{norm} \times \text{weak}^*$ closed. \diamond

Let $A : X \rightrightarrows X^*$. Following [38], we say A has the upper-semicontinuity property *property (Q)* if for every net $(x_\alpha)_{\alpha \in J}$ in X such that $x_\alpha \rightarrow x$, we have

$$(34) \quad \bigcap_{\alpha \in J} \overline{\text{conv} \left[\bigcup_{\beta \succeq_J \alpha} A(x_\beta) \right]}^{\text{w}^*} \subseteq Ax.$$

Let $A : X \rightrightarrows X^*$ be monotone with $\text{dom } A \neq \emptyset$ and consider a set $S \subseteq \text{dom } A$. We define $A_S : X \rightrightarrows X^*$ by

$$(35) \quad \begin{aligned} \text{gra } A_S &= \overline{\text{gra } A \cap (S \times X^*)}^{\|\cdot\| \times w^*} \\ &= \{(x, x^*) \mid \exists \text{ a net } (x_\alpha, x_\alpha^*)_{\alpha \in \Gamma} \text{ in } \text{gra } A \cap (S \times X^*) \text{ such that } x_\alpha \longrightarrow x, x_\alpha^* \rightharpoonup_{w^*} x^*\}. \end{aligned}$$

If $\text{int dom } A \neq \emptyset$, we denote by $A_{\text{int}} := A_{\text{int dom } A}$. We note that $\text{gra } A_{\text{dom } A} = \overline{\text{gra } A}^{\|\cdot\| \times w^*} \supseteq \text{gra } A$ while $\text{gra } A_S \subseteq \text{gra } A_T$ for $S \subseteq T$.

We now turn to consequences of these boundedness results. The following is the key technical proposition of this section.

Proposition 4.7 (See [22, Proposition 5.2].) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$ such that S is dense in $\text{int dom } A$. Assume that $x \in \text{dom } A$ and $v \in H_{\overline{\text{dom } A}}(x) = \text{int } T_{\overline{\text{dom } A}}(x)$. Then there exists $x_0^* \in A_S(x)$ such that*

$$(36) \quad \sup \langle A_S(x), v \rangle = \langle x_0^*, v \rangle = \sup \langle Ax, v \rangle.$$

In particular, $\text{dom } A_S = \text{dom } A$.

Corollary 4.8 (See [22, Corollary 5.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$. For any S dense in $\text{int dom } A$, we have $\overline{\text{conv}[A_S(x)]}^{w^*} = Ax = A_{\text{int}}(x), \forall x \in \text{int dom } A$.*

There are many possible extensions of this sort of result along the lines studied in [17]. Applying Proposition 4.7 and Lemma 4.3, we can also quickly recapture [1, Theorem 2.1].

Theorem 4.9 (Directional boundedness in Euclidean space) (See [22, Theorem 5.1].) *Suppose that X is finite-dimensional. Let $A : X \rightrightarrows X^*$ be maximally monotone and $x \in \text{dom } A$. Assume that there exist $d \in X$ and $\varepsilon_0 > 0$ such that $x + \varepsilon_0 d \in \text{int dom } A$. Then*

$$[Ax]_d := \{x^* \in Ax \mid \langle x^*, d \rangle = \sup \langle Ax, d \rangle\}$$

is nonempty and compact. Moreover, if a sequence $(x_n)_{n \in \mathbb{N}}$ in $\text{dom } A$ is such that $x_n \longrightarrow x$ and

$$(37) \quad \lim \frac{x_n - x}{\|x_n - x\|} = d,$$

then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$(38) \quad A(x_n) \subseteq [Ax]_d + \varepsilon B_{X^*}, \quad \forall n \geq N.$$

Theorem 4.10 (Reconstruction of A , I) (See [22, Theorem 5.2].) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $S \subseteq \text{int dom } A \neq \emptyset$ and with S dense in $\text{int dom } A$. Then*

$$(39) \quad Ax = N_{\overline{\text{dom } A}}(x) + \overline{\text{conv}[A_S(x)]}^{w^*}, \quad \forall x \in X.$$

Remark 4.11 (See [22, Remark 5.4].) If X is a *weak Asplund space* (as holds if X has a Gâteaux smooth equivalent norm, see [47, 50, 17]), the nets defined in A_S in Proposition 4.7 and Theorem 4.10 can be replaced by sequences. \diamond

In various classes of Banach space we can choose useful structure for $S \in S_A$, where

$$S_A := \{S \subseteq \text{int dom } A \mid S \text{ is dense in } \text{int dom } A\}.$$

By [35, 47, 50, 68, 69, 56, 21], we have multiple selections for S (see the below).

Corollary 4.12 (Specification of S_A) (See [22, Corollary 5.2].) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $\text{int dom } A \neq \emptyset$. We may choose the dense set $S \in S_A$ to be as follows:*

- (i) *In a Gâteaux smooth space, entirely within the residual set of non- σ porous points of $\text{dom } A$,*
- (ii) *In an Asplund space, to include only a subset of the generic set points of single-valuedness and norm to norm continuity of A ,*
- (iii) *In a separable Asplund space, to hold only countably many angle-bounded points of A ,*
- (iv) *In a weak Asplund space, to include only a subset of the generic set of points of single-valuedness (and norm to weak* continuity) of A ,*
- (v) *In a separable space, to include only points of single-valuedness (and norm to weak* continuity) of A whose complement is covered by a countable union of Lipschitz surfaces.*
- (vi) *In finite dimensions, to include only points of differentiability of A which are of full measure.*

These classes are sufficient but not necessary: for example, there are Asplund spaces with no equivalent Gâteaux smooth renorm [21]. Note also that in (v) and (vi) we also know that $A \setminus S$ is a null set in the senses discussed [30].

We now restrict attention to convex functions.

Corollary 4.13 (Convex subgradients) (See [22, Corollary 5.3].) *Let $f : X \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\text{int dom } f \neq \emptyset$. Let $S \subseteq \text{int dom } f$ be given with S dense in $\text{dom } f$. Then*

$$\partial f(x) = N_{\overline{\text{dom } f}}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w^*} = N_{\text{dom } f}(x) + \overline{\text{conv}[(\partial f)_S(x)]}^{w^*}, \quad \forall x \in X.$$

Remark 4.14 Results closely related to Corollary 4.13 have been obtained in [54, 3, 39, 67] and elsewhere. Interestingly, in the convex case we have obtained as much information more easily than by the direct convex analysis approach of [3]. \diamond

Now we refine Corollary 4.8 and Theorem 4.10.

Let $A : X \rightrightarrows X^*$. We define $\widehat{A} : X \rightrightarrows X^*$ by

$$(40) \quad \text{gra } \widehat{A} := \left\{ (x, x^*) \in X \times X^* \mid x^* \in \bigcap_{\varepsilon > 0} \overline{\text{conv}} [A(x + \varepsilon B_X)]^{w*} \right\}.$$

Clearly, we have $\overline{\text{gra } A}^{\|\cdot\| \times w*} \subseteq \text{gra } \widehat{A}$.

Theorem 4.15 (Reconstruction of A , II) (See [22, Theorem 5.3].) *Let $A : X \rightrightarrows X^*$ be maximally monotone with $\text{int dom } A \neq \emptyset$.*

(i) *Then $\widehat{A} = A$.*

In particular, A has property (Q); and so has a norm \times weak closed graph.*

(ii) *Moreover, if $S \subseteq \text{int dom } A$ is dense in $\text{int dom } A$ then*

$$(41) \quad \widehat{A}_S(x) := \bigcap_{\varepsilon > 0} \overline{\text{conv}} [A(S \cap (x + \varepsilon B_X))]^{w*} \supseteq \overline{\text{conv}} [A_S(x)]^{w*}, \quad \forall x \in X.$$

Thence

$$(42) \quad Ax = \widehat{A}_S(x) + N_{\overline{\text{dom } A}}(x), \quad \forall x \in X.$$

Remark 4.16 Property (Q) first introduced by Cesari in Euclidean space, was recently established for maximally monotone operators with nonempty domain interior in a barreled normed space by Voisei in [72, Theorem 42] (See also [72, Theorem 43] for the result under more general hypotheses.). Several interesting characterizations of maximally monotone operators in finite dimensional spaces, including the property (Q) were studied by Löne [40].

In general, we do not have $Ax = \overline{\text{conv}} [A_S(x)]^{w*}, \forall x \in \text{dom } A$, for a maximally monotone operator $A : X \rightrightarrows X^*$ with $S \subseteq \text{int dom } A \neq \emptyset$ such that S is dense in $\text{dom } A$.

We give a simple example to demonstrate this.

Example 4.17 (See [22, Example 6.1].) Let C be a closed convex subset of X with $S \subseteq \text{int } C \neq \emptyset$ such that S is dense in C . Then N_C is maximally monotone and $\text{gra}(N_C)_S = C \times \{0\}$, but $N_C(x) \neq \overline{\text{conv}} [(N_C)_S(x)]^{w*}, \forall x \in \text{bdry } C$. We thus must have $\bigcap_{\varepsilon > 0} \overline{\text{conv}} [N_C(x + \varepsilon B_X)]^{w*} = N_C(x), \forall x \in X$. \diamond

While the subdifferential operators in Example 4.6 necessarily fail to have property (Q), it is possible for operators with no points of continuity to possess the property. Considering any closed linear mapping A from a reflexive space X to its dual, we have $\widehat{A} = A$ and hence A has property (Q). More generally:

Example 4.18 (See [22, Example 6.2].) Suppose that X is reflexive. Let $A : X \rightrightarrows X^*$ be such that $\text{gra } A$ is nonempty closed and convex. Then $\widehat{A} = A$ and hence A has property (Q). \diamond

It would be interesting to know whether \widehat{A} and A can differ for a maximal operator with norm \times weak* closed graph.

Finally, we illustrate what Corollary 4.13 says in the case of $x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p$.

Example 4.19 (See [22, Example 6.3].) Let $p > 1$ and $f : X \rightarrow]-\infty, +\infty]$ be defined by

$$x \mapsto \iota_{B_X}(x) + \frac{1}{p}\|x\|^p.$$

Then for every $x \in \text{dom } f$, we have

$$N_{\text{dom } f}(x) = \begin{cases} \mathbb{R}_+ \cdot Jx, & \text{if } \|x\| = 1; \\ \{0\}, & \text{if } \|x\| < 1 \end{cases}, \quad (\partial f)_{\text{int}}(x) = \begin{cases} \|x\|^{p-2} \cdot Jx, & \text{if } \|x\| \neq 0; \\ \{0\}, & \text{otherwise} \end{cases}$$

where $J := \partial \frac{1}{2}\|\cdot\|^2$ and $\mathbb{R}_+ := [0, +\infty[$. Moreover, $\partial f = N_{\text{dom } f} + (\partial f)_{\text{int}} = N_{\text{dom } f} + \partial \frac{1}{p}\|\cdot\|^p$, and then $\partial f(x) \neq (\partial f)_{\text{int}}(x) = \overline{\text{conv}[(\partial f)_{\text{int}}(x)]}^{w*}, \forall x \in \text{bdry dom } f$. We also have $\bigcap_{\varepsilon > 0} \overline{\text{conv}[\partial f(x + \varepsilon B_X)]}^{w*} = \partial f(x), \forall x \in X$. \diamond

4.2 Characterizations on the domain and range of A

The following is the classical result on the convexity of the closure of the domain of a maximally monotone operator with nonempty interior domain.

Fact 4.20 (Rockafellar) (See [52, Theorem 1] or [62, Theorem 27.1 and Theorem 27.3].) *Let $A : X \rightrightarrows X^*$ be maximal monotone with $\text{int dom } A \neq \emptyset$. Then $\text{int dom } A = \text{int } \overline{\text{dom } A}$ and $\overline{\text{dom } A}$ is convex.*

The strong result below follows directly from the definition of operators of type (BR):

Proposition 4.21 (See [23, Proposition 3.5].) *Let $A : X \rightrightarrows X^*$ be maximally monotone and $(x, x^*) \in X \times X^*$. Assume that A is of type (BR) and that $\inf_{(a, a^*) \in \text{gra } A} \langle x - a, x^* - a^* \rangle > -\infty$. Then $x \in \overline{\text{dom } A}$ and $x^* \in \overline{\text{ran } A}$. In particular,*

$$\overline{\text{dom } A} = \overline{P_X[\text{dom } F_A]} \quad \text{and} \quad \overline{\text{ran } A} = \overline{P_{X^*}[\text{dom } F_A]}.$$

In particular, $\overline{\text{dom } A}$ and $\overline{\text{ran } A}$ are both convex.

We recall that every monotone operator of type (FPV) has a convex closure to its domain, while every maximally monotone continuous linear operator is of type (FPV) (see [62, Theorem 46.1] or [23]). But, Remark 2.13 shows, a maximally monotone bounded linear operator need not be of type (BR).

We turn to an interesting related result on the domain of A .

Theorem 4.22 (See [23, Theorem 3.6].) *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then*

$$\overline{\text{conv} [\text{dom } A]} = \overline{P_X [\text{dom } F_A]}.$$

Remark 4.23 Theorem 4.22 provides an affirmative answer to a question posed by Simons in [62, Problem 28.3, page 112].

Following the lines of the proof of [23, Theorem 3.6], we obtain the following counterpart result.

Theorem 4.24 *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then*

$$\overline{\text{conv} [\text{ran } A]}^{w^*} = \overline{P_{X^*} [\text{dom } F_A]}^{w^*}.$$

Proof. By Fact 3.4, it suffices to show that

$$(43) \quad P_{X^*} [\text{dom } F_A] \subseteq \overline{\text{conv} [\text{ran } A]}^{w^*}.$$

Let $(z, z^*) \in \text{dom } F_A$. We shall show that

$$(44) \quad z^* \in \overline{\text{conv} [\text{ran } A]}^{w^*}.$$

Suppose to the contrary that

$$(45) \quad z^* \notin \overline{\text{conv} [\text{ran } A]}^{w^*}.$$

Since $(z, z^*) \in \text{dom } F_A$, there exists $r \in \mathbb{R}$ such that

$$(46) \quad F_A(z, z^*) \leq r.$$

By the Separation theorem, there exist $\delta > 0$ and $y_0 \in X$ with $\|y_0\| = 1$ such that

$$(47) \quad \langle y_0, z^* - a^* \rangle > \delta, \quad \forall a^* \in \text{conv} [\text{ran } A].$$

Let $n \in \mathbb{N}$. Since $z^* \notin \overline{\text{conv} [\text{ran } A]}^{w^*}$, $(z + ny_0, z^*) \notin \text{gra } A$. By the maximal monotonicity of A , there exists $(a_n, a_n^*) \in \text{gra } A$ such that

$$(48) \quad \begin{aligned} \langle z - a_n, a_n^* - z^* \rangle &> \langle ny_0, z^* - a_n^* \rangle \Rightarrow \langle z - a_n, a_n^* - z^* \rangle > n\delta \quad (\text{by (47)}) \\ &\Rightarrow \langle z - a_n, a_n^* \rangle + \langle z^*, a_n \rangle > n\delta + \langle z, z^* \rangle. \end{aligned}$$

Then we have

$$F_A(z, z^*) \geq \sup_{n \in \mathbb{N}} \{ \langle z - a_n, a_n^* \rangle + \langle z^*, a_n \rangle \} \geq \sup_{n \in \mathbb{N}} \{ n\delta + \langle z, z^* \rangle \} = +\infty,$$

which contradicts (46). Hence $z \in \overline{\text{conv} [\text{ran } A]}^{w^*}$ and in consequence (43) holds. ■

Remark 4.25 In Theorem 4.24, we cannot replace the w^* closure by the norm closure. For Example, let A be defined as in Theorem 2.10. Theorem 4.24 implies that $\overline{\text{ran } A}^{w^*} = \overline{P_{X^*}[\text{dom } F_A]}^{w^*}$, however, $\overline{P_{X^*}[\text{dom } F_A]} \not\subseteq \overline{\text{ran } A}$ by Remark 2.11.

More concrete example is as follows. Let A_α be define as in Example 2.12. By Remark 2.13 and Theorem 4.24, $\overline{\text{ran } A_\alpha}^{w^*} = \overline{P_{X^*}[\text{dom } F_{A_\alpha}]}^{w^*}$ but $\overline{P_{X^*}[\text{dom } F_{A_\alpha}]} \not\subseteq \overline{\text{ran } A_\alpha}$.

5 Results on linear relations

This section is mainly based on the work in [6, 11, 18] by Bauschke, Borwein, Burachik, Wang and Yao. During the 1970s Brezis and Browder presented a now classical characterization of maximal monotonicity of monotone linear relations in reflexive spaces.

Theorem 5.1 (Brezis-Browder in reflexive Banach space [24, 25]) *Suppose that X is reflexive. Let $A: X \rightrightarrows X^*$ be a monotone linear relation such that $\text{gra } A$ is closed. Then A is maximally monotone if and only if the adjoint A^* is monotone.*

We extend this result in the setting of a general real Banach space. (See also [65] and [64] for Simons' recent extensions in the context of symmetrically self-dual Banach (SSDB) spaces as defined in [62, §21] and of Banach SNL spaces.)

Theorem 5.2 (Brezis-Browder in general Banach space) (See [5, Theorem 4.1].) *Let $A: X \rightrightarrows X^*$ be a monotone linear relation such that $\text{gra } A$ is closed. Then A is maximally monotone of type (D) if and only if A^* is monotone.*

This also gives an affirmative answer to a question of Phelps and Simons [49, Section 9, item 2]:

Let $A: \text{dom } A \rightarrow X^$ be linear and maximally monotone. Assume that A^* is monotone. Is A necessarily of type (D)?*

We give a corresponding but negative answer to a question posed in [77, Chapter 3.5, page 56] (see Example 5.3 below).

Let $A: X \rightrightarrows X^$ be a monotone linear relation such that $\text{gra } A$ is closed. Assume $A^*|_X$ is monotone. Is A necessarily maximally monotone?*

If Z is a real Banach space with dual Z^* and a set $S \subseteq Z$, we define S^\perp by $S^\perp := \{z^* \in Z^* \mid \langle z^*, s \rangle = 0, \forall s \in S\}$. Given a subset D of Z^* , we define D_\perp [49] by $D_\perp := \{z \in Z \mid \langle z, d^* \rangle = 0, \forall d^* \in D\}$.

Example 5.3 Let X be nonreflexive, and $e \in X^{**} \setminus X$. Let $A : X \rightrightarrows X^*$ by $\text{gra } A := \{0\} \times e_\perp$. Then A is a monotone linear relation with closed graph, and $\text{gra } A^* = \text{span}\{e\} \times X^*$. Moreover, $A^*|_X$ is monotone but A is not maximally monotone. \diamond

Proof. Clearly, A is a monotone linear relation and $\text{gra } A$ is closed. Since $e \notin X$, $e \neq 0$. Thus, $e_\perp \neq X^*$ and hence A is not maximally monotone.

Let $(z^{**}, z^*) \in X^{**} \times X^*$. Then we have

$$\begin{aligned} (z^{**}, z^*) \in \text{gra } A^{**} &\Leftrightarrow \langle z^{**}, a^* \rangle + \langle z^*, -0 \rangle = 0, \quad \forall a^* \in e_\perp \Leftrightarrow \langle z^{**}, a^* \rangle = 0, \quad \forall a^* \in e_\perp \\ &\Leftrightarrow z^{**} \in (e_\perp)^\perp \\ &\Leftrightarrow z^{**} \in \text{span}\{e\} \quad (\text{by [45, Proposition 2.6.6(c)]}). \end{aligned}$$

Hence $\text{gra } A^* = \text{span}\{e\} \times X^*$ and then $\text{gra}(A^*|_X) = \{0\} \times X^*$. Thus, $A^*|_X$ is monotone. \blacksquare

Remark 5.4 Example 5.3 gives a negative answer to a question posed in [77, Chapter 3.5, page 56]. Note that by [5, Proposition 5.4(iv)] or [77, Proposition 3.2.10(iii), page 25], the converse of [77, Chapter 3.5, page 56] is true, that is,

Let $A : X \rightrightarrows X^$ be a maximally monotone linear relation. Then $A^*|_X$ is monotone.*

Fact 5.5 (See [11, Propositions 3.5, 3.6 and 3.7 and Lemma 3.18].) *Suppose that $X = \ell^2$, and that $A : \ell^2 \rightrightarrows \ell^2$ is given by*

$$(49) \quad Ax := \frac{\left(\sum_{i < n} x_i - \sum_{i > n} x_i \right)_{n \in \mathbb{N}}}{2} = \left(\sum_{i < n} x_i + \frac{1}{2} x_n \right)_{n \in \mathbb{N}}, \quad \forall x = (x_n)_{n \in \mathbb{N}} \in \text{dom } A,$$

where $\text{dom } A := \left\{ x := (x_n)_{n \in \mathbb{N}} \in \ell^2 \mid \sum_{i \geq 1} x_i = 0, \left(\sum_{i \leq n} x_i \right)_{n \in \mathbb{N}} \in \ell^2 \right\}$ and $\sum_{i < 1} x_i := 0$. Then

$$(50) \quad A^*x = \left(\frac{1}{2} x_n + \sum_{i > n} x_i \right)_{n \in \mathbb{N}},$$

where

$$x = (x_n)_{n \in \mathbb{N}} \in \text{dom } A^* = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \ell^2 \mid \left(\sum_{i > n} x_i \right)_{n \in \mathbb{N}} \in \ell^2 \right\}.$$

Then A provides an at most single-valued linear relation such that the following hold.

- (i) A is maximally monotone and skew.
- (ii) A^* is maximally monotone but not skew.
- (iii) $F_{A^*}^*(x^*, x) = F_{A^*}(x, x^*) = \iota_{\text{gra } A^*}(x, x^*) + \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X.$

(iv) $\langle A^*x, x \rangle = \frac{1}{2}s^2$, $\forall x = (x_n)_{n \in \mathbb{N}} \in \text{dom } A^*$ with $s := \sum_{i \geq 1} x_i$.

Let $F : X \times X^* \rightarrow]-\infty, +\infty]$, and define $\text{pos } F$ [62] by

$$\text{pos } F := \{(x, x^*) \in X \times X^* \mid F(x, x^*) = \langle x, x^* \rangle\}.$$

The following result due to Simons generalizes the result of Brézis, Crandall and Pazy [26].

Fact 5.6 (Simons) (See [62, Theorem 34.3].) *Suppose that X is reflexive. Let $F_1, F_2 : X \times X^* \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous and convex functions with $P_X \text{dom } F_1 \cap P_X \text{dom } F_2 \neq \emptyset$. Assume that F_1, F_2 are BC-functions and that there exists an increasing function $j : [0, +\infty[\rightarrow [0, +\infty[$ such that the implication*

$$\begin{aligned} (x, x^*) \in \text{pos } F_1, (y, y^*) \in \text{pos } F_2, x \neq y \text{ and } \langle x - y, y^* \rangle &= \|x - y\| \cdot \|y^*\| \\ \Rightarrow \|y^*\| &\leq j(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|) \end{aligned}$$

holds. Then $M := \{(x, x^ + y^*) \mid (x, x^*) \in \text{pos } F_1, (x, y^*) \in \text{pos } F_2\}$ is a maximally monotone set.*

Example 5.7 (See [8, Example 5.2].) Suppose that X and A are as in Fact 5.5. Set $e_1 := (1, 0, \dots, 0, \dots)$, i.e., there is a 1 in the first place and all others entries are 0, and $C := [0, e_1]$. Let $j : [0, +\infty[\rightarrow [0, +\infty[$ be an increasing function such that $j(\gamma) \geq \frac{\gamma}{2}$ for every $\gamma \in [0, +\infty[$. Then the following hold.

(i) F_{A^*} and $F_{N_C} = \iota_C \oplus \sigma_C$ are BC-functions.

(ii) $(F_{A^*} \square_2 F_{N_C})(x, x^*) = \begin{cases} \langle x, A^*x \rangle + \sigma_C(x^* - A^*x), & \text{if } x \in C; \\ +\infty, & \text{otherwise,} \end{cases} \quad \forall (x, x^*) \in X \times X^*.$

(iii) Then

$$F_{A^*}^*(x^*, 0) + F_{N_C}^*(A^*e_1 - x^*, 0) > (F_{A^*} \square_2 F_{N_C})^*(A^*e_1, 0), \quad \forall x^* \in X.$$

(iv) The implication

$$\begin{aligned} (x, x^*) \in \text{pos } F_{N_C}, (y, y^*) \in \text{pos } F_{A^*}, x \neq y \text{ and } \langle x - y, y^* \rangle &= \|x - y\| \cdot \|y^*\| \\ \Rightarrow \|y^*\| &\leq \frac{1}{2}\|y\| \leq j(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|) \end{aligned}$$

holds.

(v) $A^* + N_C$ is maximally monotone. ◇

Example 5.7 shows the following conjecture fails in general (see [8] for more on Open Problem 5.8).

Open Problem 5.8 Suppose that X is reflexive. Let $F_1, F_2 : X \times X^* \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous and convex functions with $P_X \operatorname{dom} F_1 \cap P_X \operatorname{dom} F_2 \neq \emptyset$. Assume that F_1, F_2 are BC-functions and that there exists an increasing function $j : [0, +\infty[\rightarrow [0, +\infty[$ such that the implication

$$\begin{aligned} (x, x^*) \in \operatorname{pos} F_1, (y, y^*) \in \operatorname{pos} F_2, x \neq y \text{ and } \langle x - y, y^* \rangle &= \|x - y\| \cdot \|y^*\| \\ \Rightarrow \|y^*\| &\leq j(\|x\| + \|x^* + y^*\| + \|y\| + \|x - y\| \cdot \|y^*\|) \end{aligned}$$

holds. Then, is it true that, for all $(z, z^*) \in X \times X^*$, there exists $v^* \in X^*$ such that

$$(51) \quad F_1^*(v^*, z) + F_2^*(z^* - v^*, z) \leq (F_1 \square_2 F_2)^*(z^*, z)?$$

Finally, we provide some results on the partial inf-convolution of two Fitzpatrick functions associated with maximally monotone operators, which has important consequences for the “sum problem” (see the discussion in Section 6).

Proposition 5.9 (See [77, Proposition 7.1.11, page 164].) *Let $A, B : X \rightrightarrows X^*$ be maximally monotone and suppose that $\bigcup_{\lambda > 0} \lambda [\operatorname{dom} A - \operatorname{dom} B]$ is a closed subspace of X . Then $F_A \square_2 F_B$ is proper, $\operatorname{norm} \times \operatorname{weak}^*$ lower semicontinuous and convex, and the partial infimal convolution is exact everywhere.*

Theorem 5.10 below was proved in [10, Theorem 5.10] for a reflexive space. It can be extended to a general Banach space.

Theorem 5.10 (Fitzpatrick function of the sum) (See [18, Theorem 5.2].) *Let $A, B : X \rightrightarrows X^*$ be maximally monotone linear relations, and suppose that $\operatorname{dom} A - \operatorname{dom} B$ is closed. Then*

$$F_{A+B} = F_A \square_2 F_B,$$

and the partial infimal convolution is exact everywhere.

Theorem 5.10 provides a new approach to showing the maximal monotonicity of two maximally monotone linear relations (see [18, Theorem 5.5]), which was first used by Voisei in [70] while Simons gave another proof in [62, Theorem 46.3].

Theorem 5.11 (See [18, Theorem 5.5].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Suppose C is a nonempty closed convex subset of X , and that $\operatorname{dom} A \cap \operatorname{int} C \neq \emptyset$. Then $F_{A+N_C} = F_A \square_2 F_{N_C}$, and the partial infimal convolution is exact everywhere.*

6 Open problems in Monotone Operator Theory

As discussed in [12, 13, 15, 21], the two most central open questions in monotone operator theory in a general real Banach space are almost certainly the following:

Let $A, B : X \rightrightarrows X^*$ be maximally monotone.

- (i) The “sum problem”: Assume that A, B satisfy *Rockafellar’s constraint qualification*, i.e., $\text{dom } A \cap \text{int dom } B \neq \emptyset$ [55]. *Is the sum operator $A + B$ necessarily maximally monotone*, which is so called the “sum problem”?
- (ii) *Is $\overline{\text{dom } A}$ necessarily convex?* Rockafellar showed that it is true for every operator with nonempty interior domain [52] and as we saw in Section 4.2 it is now known to hold for most classes of maximally monotone operators (see also [62, Section 44]).

A positive answer to various restricted versions of (i) implies a positive answer to (ii) [21, 62]. Some recent developments on the sum problem can be found in Simons’ monograph [62] and [12, 13, 15, 21, 75, 44, 76, 78]. Recent progress regarding (ii) can be found in [23].

In the following, we show that one possible approach to the sum problem cannot be feasible. By [62, Lemma 23.9] or [9, Proposition 4.2], $F_A \square_2 F_B \geq F_{A+B}$. It naturally raises a question: Does equality always hold under the Rockafellar’s constraint qualification? If this were true, then it would directly solve the sum problem in the affirmative (see [71, 62] and [77, Chapter 7]). However, in general, it cannot hold. The easiest example probably is [9, Example 4.7] by Bauschke, McLaren and Sendov using two projection operators on one dimensional space.

Here we give another counterexample of a a maximally monotone linear relation and the subdifferential of a proper lower semicontinuous sublinear function, which thus also implies that we cannot establish the maximality of the sum of a linear relation A and the subdifferential of a proper lower semicontinuous sublinear function f by showing that $F_A \square_2 F_{\partial f} = F_{A+\partial f}$ always holds.

Example 6.1 (See [77, Example 7.1.14, page 167].) Let X be a Hilbert space, B_X be the closed unit ball of X and Id be the identity mapping from X to X . Let $f : x \in X \rightarrow \|x\|$. Then we have

$$(52) \quad F_{\partial f} \square_2 F_{\text{Id}}(x, x^*) = \|x\| + \begin{cases} 0, & \text{if } \|x + x^*\| \leq 1; \\ \frac{1}{4}\|x + x^*\|^2 - \frac{1}{2}\|x + x^*\| + \frac{1}{4}, & \text{if } \|x + x^*\| > 1. \end{cases}$$

We also have $F_{\partial f + \text{Id}} \neq F_{\partial f} \square_2 F_{\text{Id}}$ when $X = \mathbb{R}$. ◇

Now we show another one possible approach to the sum problem cannot be feasible either.

Let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be proper lower semicontinuous and convex. Assume that

$$F(x, x^*) \geq \langle x, x^* \rangle, \quad F^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.$$

Is $\text{pos } F$ a maximally monotone set?

If the above conjecture is true, then the sum problem would have an affirmative answer by setting $F := F_A \square_2 F_B$. Burachik and Svaiter showed the conjecture holds when X is reflexive (see [29, Theorem 3.1] or [66, Theorem 1.4(b)]). We give the following example to show that it cannot be true in a general Banach space.

Example 6.2 Let X be nonreflexive, and $e \in X^{**} \setminus X$. Let $F : X \times X^* \rightarrow]-\infty, +\infty]$ be defined by $F := \iota_{\{0\} \times e_\perp}$. Then $F^* = \iota_{X^* \times \text{span}\{e\}}$ on $X^* \times X^{**}$, and

$$F(x, x^*) \geq \langle x, x^* \rangle, \quad F^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.$$

However, $\text{pos } F = \{0\} \times e_\perp$ is not a maximally monotone set. \diamond

Proof. We have F is proper lower semicontinuous and convex. Similar to the proof of Example 5.3, we have $F^* = \sigma_{\{0\} \times e_\perp} = \iota_{X^* \times \text{span}\{e\}}$. Then we have $F \geq \langle \cdot, \cdot \rangle$, $F^{*\top} \geq \langle \cdot, \cdot \rangle$ on $X \times X^*$. Clearly, $\text{pos } F = \{0\} \times e_\perp$. By Example 5.3, it is not a maximally monotone set. \blacksquare

Remark 6.3 (Conjecture) Finally, we conjecture that every nonreflexive space admits an operator that is not of type (BR) and so also not of type (D).

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